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Note

On Smoothness and Proximal Points*

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A normed linear space X is said to have property (P) if for every pair of convex sets U and V in X, u nearest to v from U and v nearest to u from V implies $\| u - v \| = \inf\{\| u' - v' \| : u' \in U, v' \in V\}$. If (u, v) satisfies this last condition, the pair is called proximal. A recent paper of Pai [1] demonstrates that X is smooth if and only if X has property P. The purpose of this note is to supply a short, geometrically appealing proof of this fact using only a standard separation theorem

THEOREM. *X is smooth if and only if X has property* (P).

Proof. Suppose X is smooth U, V, u, v satisfy the conditions in the hypothesis of (P). Let B_u and B_v be closed balls, radius $\| u - v \|$, with centers u and v, respectively. Then there are hyperplanes H_u and H_v such that H_u separates B_u and V and H_v separates B_v and U. Then H_u is the unique hyperplane of support to B_u at v and H_v is the unique hyperplane of support to B_v at u. Since any ball in X is symmetric with respect to its center, it follows immediately that H_u and H_v are parallel. Let $H_u{}^+$ and $H_v{}^+$ be closed half-spaces determined by the H_u and H_v, respectively, and containing V and U, respectively. Then $u' \in U, v' \in V$ imply $u' \in H_v{}^+$ and $v' \in H_u{}^+$. Hence $\| u' - v' \| \geqslant \| u - v \|$ and (u, v) is a proximal pair.

Conversely, suppose X is not smooth and that x is a point on the unit ball where there are 2 hyperplanes of support, H_1 and H_1^*. Let H_2 be a hyperplane of support to the ball, center x, radius 1, at 0. By symmetry, we may choose H_2 parallel to H_1^*. Hence $H_1 \cap H_2 \neq \varnothing$. Let $U = H_2{}^+$, the closed half-space determined by H_2 and not containing x. Let $V = H_1{}^+$, the closed half-space determined by H_1 and not containing 0. These sets are convex, x is

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nearest to 0 from V and 0 is nearest to x from U. $\| x \| = 1$ but $H_1 \cap H_2 \neq \varnothing$ implies $\inf\{\| u' - v' \| : u' \in U, v' \in V\} = 0$. So X cannot have property (P).

Reference

1. D. V. Pai, A characterization of smooth normed linear spaces, *J. Approximation Theory* **17** (1976), 315–320.

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